

Estimates for the number of vertices with an interval spectrum in proper edge colorings of some graphs

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A proper edge t -coloring of a graph G is a coloring of edges of G with colors $1, 2, \dots, t$ such that each of t colors is used, and adjacent edges are colored differently. The set of colors of edges incident with a vertex x of G is called a spectrum of x . A proper edge t -coloring of a graph G is interval for its vertex x if the spectrum of x is an interval of integers. A proper edge t -coloring of a graph G is persistent-interval for its vertex x if the spectrum of x is an interval of integers beginning from the color 1.

For graphs G from some classes of graphs, we obtain estimates for the possible number of vertices for which a proper edge t -coloring of G can be interval or persistent-interval.

1. Introduction

We consider undirected, simple, finite, connected graphs. For a graph G , we denote by $V(G)$ and $E(G)$ the sets of its vertices and edges, respectively. For any $x \in V(G)$, $d_G(x)$ denotes the degree of the vertex x in G . For a graph G , we denote by $\Delta(G)$ the maximum degree of a vertex of G . A function $\varphi : E(G) \rightarrow \{1, 2, \dots, t\}$ is called a proper edge t -coloring of a graph G if each of t colors is used, and adjacent edges are colored differently. The set of all proper edge t -colorings of G is denoted by $\alpha(G, t)$. The minimum value of t for which there exists a proper edge t -coloring of a graph G is called a chromatic index [22] of G and is denoted by $\chi'(G)$. Let us also define the set $\alpha(G)$ of all proper edge colorings of the graph G

$$\alpha(G) \equiv \bigcup_{t=\chi'(G)}^{|E(G)|} \alpha(G, t).$$

If G is a graph, $\varphi \in \alpha(G)$, $x \in V(G)$, then the set of colors of edges of G incident with x is called a spectrum of the vertex x in the coloring φ of the graph G and is denoted by $S_G(x, \varphi)$.

An arbitrary nonempty subset of consecutive integers is called an interval. An interval with the minimum element p and the maximum element q is denoted by $[p, q]$. An interval D is called a h -interval if $|D| = h$.

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For any real number ξ , we denote by $\lfloor \xi \rfloor$ ($\lceil \xi \rceil$) the maximum (minimum) integer which is less (greater) than or equal to ξ .

If G is a graph, $\varphi \in \alpha(G)$, and $x \in V(G)$, then we say that φ is interval (persistent-interval) for x if $S_G(x, \varphi)$ is a $d_G(x)$ -interval (a $d_G(x)$ -interval with 1 as its minimum element). For an arbitrary graph G and any $\varphi \in \alpha(G)$, we denote by $f_{G,i}(\varphi)$ ($f_{G,pi}(\varphi)$) the number of vertices of the graph G for which φ is interval (persistent-interval). For any graph G , let us [17] set

$$\eta_i(G) \equiv \max_{\varphi \in \alpha(G)} f_{G,i}(\varphi), \quad \eta_{pi}(G) \equiv \max_{\varphi \in \alpha(G)} f_{G,pi}(\varphi).$$

A bipartite graph G with bipartition (X, Y) is called (a, b) -biregular, if $d_G(x) = a$ for any vertex $x \in X$, and $d_G(y) = b$ for any vertex $y \in Y$.

The terms and concepts that we do not define can be found in [23].

It is clear that if for any graph G $\eta_{pi}(G) = |V(G)|$, then $\chi'(G) = \Delta(G)$. For a regular graph G , these two conditions are equivalent: $\eta_{pi}(G) = |V(G)|$ iff $\chi'(G) = \Delta(G)$. It is known [15, 19] that for a regular graph G , the problem of deciding whether $\chi'(G) = \Delta(G)$ or not is NP -complete. It means that for a regular graph G , the problem of deciding whether $\eta_{pi}(G) = |V(G)|$ or not is also NP -complete. For any tree G , some necessary and sufficient condition for $\eta_{pi}(G) = |V(G)|$ was obtained in [8]. In this paper, for an arbitrary regular graph G , we obtain a lower bound for the parameter $\eta_{pi}(G)$.

If G is a graph, $R_0 \subseteq V(G)$, and the coloring $\varphi \in \alpha(G)$ is interval (persistent-interval) for any $x \in R_0$, then we say that φ is interval (persistent-interval) on R_0 .

$\varphi \in \alpha(G)$ is called an interval coloring of a graph G if φ is interval on $V(G)$.

We define the set \mathfrak{N} as the set of all graphs for which there is an interval coloring. Clearly, for any graph G , $G \in \mathfrak{N}$ if and only if $\eta_i(G) = |V(G)|$.

The notion of an interval coloring was introduced in [6]. In [6, 16, 7] it is shown that if $G \in \mathfrak{N}$, then $\chi'(G) = \Delta(G)$. For a regular graph G , these two conditions are equivalent: $G \in \mathfrak{N}$ iff $\chi'(G) = \Delta(G)$ [6, 16, 7]. Consequently, for a regular graph G , four conditions are equivalent: $G \in \mathfrak{N}$, $\chi'(G) = \Delta(G)$, $\eta_i(G) = |V(G)|$, $\eta_{pi}(G) = |V(G)|$. It means that for any regular graph G ,

1. the problem of deciding whether or not G has an interval coloring is NP -complete,
2. the problem of deciding whether $\eta_i(G) = |V(G)|$ or not is NP -complete.

In this paper, for an arbitrary regular graph G , we obtain a lower bound for the parameter $\eta_i(G)$.

We also obtain some results for bipartite graphs. The complexity of the problem of existence of an interval coloring for bipartite graphs is investigated in [3, 9, 21]. In [16] it is shown that for a bipartite graph G with bipartition (X, Y) and $\Delta(G) = 3$ the problem of existence of a proper edge 3-coloring which is persistent-interval on $X \cup Y$ (or even only on Y [6, 16]) is NP -complete.

Suppose that G is an arbitrary bipartite graph with bipartition (X, Y) . Then $\eta_i(G) \geq \max\{|X|, |Y|\}$.

Suppose that G is a bipartite graph with bipartition (X, Y) for which there exists a coloring $\varphi \in \alpha(G)$ persistent-interval on Y . Then $\eta_{pi}(G) \geq 1 + |Y|$.

Some attention is devoted to (a, b) -biregular bipartite graphs [4, 14, 13, 18] in the case $b = a + 1$.

We show that if G is a $(k - 1, k)$ -biregular bipartite graph, $k \geq 4$, then

$$\eta_i(G) \geq \frac{k-1}{2k-1} \cdot |V(G)| + \left\lceil \frac{k}{\left\lceil \frac{k}{2} \right\rceil \cdot (2k-1)} \cdot |V(G)| \right\rceil.$$

We show that if G is a $(k - 1, k)$ -biregular bipartite graph, $k \geq 3$, then

$$\eta_{pi}(G) \geq \frac{k}{2k-1} \cdot |V(G)|.$$

2. Results

Theorem 1 [17] *If G is a regular graph with $\chi'(G) = 1 + \Delta(G)$, then*

$$\eta_{pi}(G) \geq \left\lceil \frac{|V(G)|}{1 + \Delta(G)} \right\rceil.$$

Proof. Suppose that $\beta \in \alpha(G, 1 + \Delta(G))$. For any $j \in [1, 1 + \Delta(G)]$, define

$$V_{G,\beta,j} \equiv \{x \in V(G) / j \notin S_G(x, \beta)\}.$$

For arbitrary integers j', j'' , where $1 \leq j' < j'' \leq 1 + \Delta(G)$, we have

$$V_{G,\beta,j'} \cap V_{G,\beta,j''} = \emptyset$$

and

$$\bigcup_{j=1}^{1+\Delta(G)} V_{G,\beta,j} = V(G).$$

Hence, there exists $j_0 \in [1, 1 + \Delta(G)]$ for which

$$|V_{G,\beta,j_0}| \geq \left\lceil \frac{|V(G)|}{1 + \Delta(G)} \right\rceil.$$

Set $R_0 \equiv V_{G,\beta,j_0}$.

Case 1 $1j_0 = 1 + \Delta(G)$.

Clearly, β is persistent-interval on R_0 .

Case 2 $2j_0 \in [1, \Delta(G)]$.

Define a function $\varphi : E(G) \rightarrow [1, 1 + \Delta(G)]$. For any $e \in E(G)$, set:

$$\varphi(e) \equiv \begin{cases} \beta(e), & \text{if } \beta(e) \notin \{j_0, 1 + \Delta(G)\} \\ j_0, & \text{if } \beta(e) = 1 + \Delta(G) \\ 1 + \Delta(G), & \text{if } \beta(e) = j_0. \end{cases}$$

It is not difficult to see that $\varphi \in \alpha(G, 1 + \Delta(G))$ and φ is persistent-interval on R_0 .

Corollary 1 [17] *If G is a cubic graph, then there exists a coloring from $\alpha(G, \chi'(G))$ which is persistent-interval for at least $\left\lceil \frac{|V(G)|}{4} \right\rceil$ vertices of G .*

Theorem 2 [17] *If G is a regular graph with $\chi'(G) = 1 + \Delta(G)$, then*

$$\eta_i(G) \geq \left\lceil \frac{|V(G)|}{\left\lceil \frac{1+\Delta(G)}{2} \right\rceil} \right\rceil.$$

Proof. Suppose that $\beta \in \alpha(G, 1 + \Delta(G))$. For any $j \in [1, 1 + \Delta(G)]$, define

$$V_{G,\beta,j} \equiv \{x \in V(G) / j \notin S_G(x, \beta)\}.$$

For arbitrary integers j', j'' , where $1 \leq j' < j'' \leq 1 + \Delta(G)$, we have

$$V_{G,\beta,j'} \cap V_{G,\beta,j''} = \emptyset$$

and

$$\bigcup_{j=1}^{1+\Delta(G)} V_{G,\beta,j} = V(G).$$

For any $i \in [1, \left\lceil \frac{1+\Delta(G)}{2} \right\rceil]$, let us define the subset $V(G, \beta, i)$ of the set $V(G)$ as follows:

$$V(G, \beta, i) \equiv \begin{cases} V_{G,\beta,2i-1} \cup V_{G,\beta,2i}, & \text{if } \Delta(G) \text{ is odd and } i \in [1, \frac{1+\Delta(G)}{2}] \\ & \text{or } \Delta(G) \text{ is even and } i \in [1, \frac{\Delta(G)}{2}], \\ V_{G,\beta,1+\Delta(G)}, & \text{if } \Delta(G) \text{ is even and } i = 1 + \frac{\Delta(G)}{2}. \end{cases}$$

For arbitrary integers i', i'' , where $1 \leq i' < i'' \leq \left\lceil \frac{1+\Delta(G)}{2} \right\rceil$, we have

$$V(G, \beta, i') \cap V(G, \beta, i'') = \emptyset$$

and

$$\bigcup_{i=1}^{\left\lceil \frac{1+\Delta(G)}{2} \right\rceil} V(G, \beta, i) = V(G).$$

Hence, there exists $i_0 \in [1, \left\lceil \frac{1+\Delta(G)}{2} \right\rceil]$ for which

$$|V(G, \beta, i_0)| \geq \left\lceil \frac{|V(G)|}{\left\lceil \frac{1+\Delta(G)}{2} \right\rceil} \right\rceil.$$

Set $R_0 \equiv V(G, \beta, i_0)$.

Case 3 1 $i_0 = \left\lceil \frac{1+\Delta(G)}{2} \right\rceil$.

Case 4 1.a $\Delta(G)$ is even.

Clearly, β is interval on R_0 .

Case 5 1.b $\Delta(G)$ is odd.

Define a function $\varphi : E(G) \rightarrow [1, 1 + \Delta(G)]$. For any $e \in E(G)$, set:

$$\varphi(e) \equiv \begin{cases} (\beta(e) + 1)(\bmod(1 + \Delta(G))), & \text{if } \beta(e) \neq \Delta(G), \\ 1 + \Delta(G), & \text{if } \beta(e) = \Delta(G). \end{cases}$$

It is not difficult to see that $\varphi \in \alpha(G, 1 + \Delta(G))$ and φ is interval on R_0 .

Case 6 2 $1 \leq i_0 \leq \lceil \frac{\Delta(G)-1}{2} \rceil$.

Define a function $\varphi : E(G) \rightarrow [1, 1 + \Delta(G)]$. For any $e \in E(G)$, set:

$$\varphi(e) \equiv \begin{cases} (\beta(e) + 2 + \Delta(G) - 2i_0)(\bmod(1 + \Delta(G))), & \text{if } \beta(e) \neq 2i_0 - 1, \\ 1 + \Delta(G), & \text{if } \beta(e) = 2i_0 - 1. \end{cases}$$

It is not difficult to see that $\varphi \in \alpha(G, 1 + \Delta(G))$ and φ is interval on R_0 .

Corollary 2 [17] If G is a cubic graph, then there exists a coloring from $\alpha(G, \chi'(G))$ which is interval for at least $\frac{|V(G)|}{2}$ vertices of G .

Theorem 3 [6, 16, 7] Let G be a bipartite graph with bipartition (X, Y) . Then there exists a coloring $\varphi \in \alpha(G, |E(G)|)$ which is interval on X .

Corollary 3 Let G be a bipartite graph with bipartition (X, Y) . Then $\eta_i(G) \geq \max\{|X|, |Y|\}$.

Theorem 4 [1, 6, 7] Let G be a bipartite graph with bipartition (X, Y) where $d_G(x) \leq d_G(y)$ for each edge $(x, y) \in E(G)$ with $x \in X$ and $y \in Y$. Then there exists a coloring $\varphi_0 \in \alpha(G, \Delta(G))$ which is persistent-interval on Y .

Theorem 5 Suppose G is a bipartite graph with bipartition (X, Y) , and there exists a coloring $\varphi_0 \in \alpha(G, \Delta(G))$ which is persistent-interval on Y . Then, for an arbitrary vertex $x_0 \in X$, there exists $\psi \in \alpha(G, \Delta(G))$ which is persistent-interval on $\{x_0\} \cup Y$.

Proof. **Case 7** 1 $S_G(x_0, \varphi_0) = [1, d_G(x_0)]$. In this case ψ is φ_0 .

Case 8 2 $S_G(x_0, \varphi_0) \neq [1, d_G(x_0)]$.

Clearly, $[1, d_G(x_0)] \setminus S_G(x_0, \varphi_0) \neq \emptyset$, $S_G(x_0, \varphi_0) \setminus [1, d_G(x_0)] \neq \emptyset$. Since $|S_G(x_0, \varphi_0)| = |[1, d_G(x_0)]| = d_G(x_0)$, there exists $\nu_0 \in [1, d_G(x_0)]$ satisfying the condition $|[1, d_G(x_0)] \setminus S_G(x_0, \varphi_0)| = |S_G(x_0, \varphi_0) \setminus [1, d_G(x_0)]| = \nu_0$.

Now let us construct the sequence $\Theta_0, \Theta_1, \dots, \Theta_{\nu_0}$ of proper edge $\Delta(G)$ -colorings of the graph G , where for any $i \in [0, \nu_0]$, Θ_i is persistent-interval on Y .

Set $\Theta_0 \equiv \varphi_0$.

Suppose that for some $k \in [0, \nu_0 - 1]$, the subsequence $\Theta_0, \Theta_1, \dots, \Theta_k$ is already constructed.

Let

$$\begin{aligned} t_k &\equiv \max(S_G(x_0, \Theta_k) \setminus [1, d_G(x_0)]), \\ s_k &\equiv \min([1, d_G(x_0)] \setminus S_G(x_0, \Theta_k)). \end{aligned}$$

Clearly, $t_k > s_k$. Consider the path $P(k)$ in the graph G of maximum length with the initial vertex x_0 whose edges are alternatively colored by the colors t_k and s_k . Let Θ_{k+1} is obtained from Θ_k by interchanging the two colors t_k and s_k along $P(k)$.

It is not difficult to see that Θ_{ν_0} is persistent-interval on $\{x_0\} \cup Y$. Set $\psi \equiv \Theta_{\nu_0}$.

Corollary 4 *Let G be a bipartite graph with bipartition (X, Y) where $d_G(x) \leq d_G(y)$ for each edge $(x, y) \in E(G)$ with $x \in X$ and $y \in Y$. Let x_0 be an arbitrary vertex of X . Then there exists a coloring $\varphi_0 \in \alpha(G, \Delta(G))$ which is persistent-interval on $\{x_0\} \cup Y$.*

Corollary 5 [17] *Let G be a bipartite graph with bipartition (X, Y) where $d_G(x) \leq d_G(y)$ for each edge $(x, y) \in E(G)$ with $x \in X$ and $y \in Y$. Then $\eta_{pi}(G) \geq 1 + |Y|$.*

Remark 1 *Notice that the complete bipartite graph $K_{n+1,n}$ for an arbitrary positive integer n satisfies the conditions of Corollary 5. It is not difficult to see that $\eta_{pi}(K_{n+1,n}) = 1 + n$. It means that the bound obtained in Corollary 5 is sharp since in this case $|Y| = n$.*

Remark 2 *Let G be a bipartite $(k-1, k)$ -biregular graph with bipartition (X, Y) , where $k \geq 3$. Then the numbers $\frac{|X|}{k}$, $\frac{|Y|}{k-1}$, and $\frac{|V(G)|}{2k-1}$ are integer. It follows from the equalities $\gcd(k-1, k) = 1$ and $|E(G)| = |X| \cdot (k-1) = |Y| \cdot k$.*

Theorem 6 [17] *Let G be a bipartite $(k-1, k)$ -biregular graph, where $k \geq 4$. Then*

$$\eta_i(G) \geq \frac{k-1}{2k-1} \cdot |V(G)| + \left\lceil \frac{k}{\lceil \frac{k}{2} \rceil \cdot (2k-1)} \cdot |V(G)| \right\rceil.$$

Proof. Suppose that (X, Y) is a bipartition of G . Clearly, $\chi'(G) = \Delta(G) = k$. Suppose that $\beta \in \alpha(G, k)$. For any $j \in [1, k]$, define:

$$V_{G,\beta,j} \equiv \{x \in X / j \notin S_G(x, \beta)\}.$$

For arbitrary integers j', j'' , where $1 \leq j' < j'' \leq k$, we have

$$V_{G,\beta,j'} \cap V_{G,\beta,j''} = \emptyset$$

and

$$\bigcup_{j=1}^k V_{G,\beta,j} = X.$$

For any $i \in [1, \lceil \frac{k}{2} \rceil]$, let us define the subset $V(G, \beta, i)$ of the set X as follows:

$$V(G, \beta, i) \equiv \begin{cases} V_{G,\beta,2i-1} \cup V_{G,\beta,2i}, & \text{if } k \text{ is odd and } i \in [1, \frac{k-1}{2}] \\ & \text{or } k \text{ is even and } i \in [1, \frac{k}{2}], \\ V_{G,\beta,k}, & \text{if } k \text{ is odd and } i = \frac{1+k}{2}. \end{cases}$$

For arbitrary integers i', i'' , where $1 \leq i' < i'' \leq \lceil \frac{k}{2} \rceil$, we have

$$V(G, \beta, i') \cap V(G, \beta, i'') = \emptyset$$

and

$$\bigcup_{i=1}^{\lceil \frac{k}{2} \rceil} V(G, \beta, i) = X.$$

Hence, there exists $i_0 \in [1, \lceil \frac{k}{2} \rceil]$ for which

$$|V(G, \beta, i_0)| \geq \left\lceil \frac{|X|}{\lceil \frac{k}{2} \rceil} \right\rceil.$$

Set $R_0 \equiv Y \cup V(G, \beta, i_0)$.

It is not difficult to verify that

$$|R_0| \geq \frac{k-1}{2k-1} \cdot |V(G)| + \left\lceil \frac{k}{\lceil \frac{k}{2} \rceil \cdot (2k-1)} \cdot |V(G)| \right\rceil.$$

Case 9 1 $i_0 = \lceil \frac{k}{2} \rceil$.

Case 10 1.a k is odd.

Clearly, β is interval on R_0 .

Case 11 1.b k is even.

Define a function $\varphi : E(G) \rightarrow [1, k]$. For any $e \in E(G)$, set:

$$\varphi(e) \equiv \begin{cases} (\beta(e) + 1)(\text{mod } k), & \text{if } \beta(e) \neq k-1, \\ k, & \text{if } \beta(e) = k-1. \end{cases}$$

It is not difficult to see that $\varphi \in \alpha(G, k)$ and φ is interval on R_0 .

Case 12 2 $i_0 \in [1, \lceil \frac{k}{2} \rceil - 1]$.

Define a function $\varphi : E(G) \rightarrow [1, k]$. For any $e \in E(G)$, set:

$$\varphi(e) \equiv \begin{cases} (\beta(e) + 1 + k - 2i_0)(\text{mod } k), & \text{if } \beta(e) \neq 2i_0 - 1, \\ k, & \text{if } \beta(e) = 2i_0 - 1. \end{cases}$$

It is not difficult to see that $\varphi \in \alpha(G, k)$ and φ is interval on R_0 .

Corollary 6 [17] Let G be a bipartite $(k-1, k)$ -biregular graph, where k is even and $k \geq 4$. Then

$$\eta_i(G) \geq \frac{k+1}{2k-1} \cdot |V(G)|.$$

Corollary 7 [17] Let G be a bipartite $(3, 4)$ -biregular graph. Then there exists a coloring from $\alpha(G, 4)$ which is interval for at least $\frac{5}{7}|V(G)|$ vertices of G .

Remark 3 For an arbitrary bipartite graph G with $\Delta(G) \leq 3$, there exists an interval coloring of G [12, 10, 11]. Consequently, if G is a bipartite $(2, 3)$ -biregular graph, then $\eta_i(G) = |V(G)|$.

Remark 4 Some sufficient conditions for existence of an interval coloring of a $(3, 4)$ -biregular bipartite graph were obtained in [2, 5, 20].

Theorem 7 [17] *Let G be a bipartite $(k-1, k)$ -biregular graph, where $k \geq 3$. Then*

$$\eta_{pi}(G) \geq \frac{k}{2k-1} \cdot |V(G)|.$$

Proof. Suppose that (X, Y) is a bipartition of G . Clearly, $\chi'(G) = \Delta(G) = k$. Suppose that $\beta \in \alpha(G, k)$.

For any $j \in [1, k]$, define:

$$V_{G,\beta,j} \equiv \{x \in X/j \notin S_G(x, \beta)\}.$$

For arbitrary integers j', j'' , where $1 \leq j' < j'' \leq k$, we have

$$V_{G,\beta,j'} \cap V_{G,\beta,j''} = \emptyset$$

and

$$\bigcup_{j=1}^k V_{G,\beta,j} = X.$$

Hence, there exists $j_0 \in [1, k]$ for which

$$|V_{G,\beta,j_0}| \geq \frac{|X|}{k}.$$

Set $R_0 \equiv Y \cup V_{G,\beta,j_0}$.

It is not difficult to verify that

$$|R_0| \geq \frac{k}{2k-1} \cdot |V(G)|.$$

Case 13 1 $j_0 = k$.

Clearly, β is persistent-interval on R_0 .

Case 14 2 $j_0 \in [1, k-1]$.

Define a function $\varphi : E(G) \rightarrow [1, k]$. For any $e \in E(G)$, set:

$$\varphi(e) \equiv \begin{cases} \beta(e), & \text{if } \beta(e) \notin \{j_0, k\} \\ j_0, & \text{if } \beta(e) = k \\ k, & \text{if } \beta(e) = j_0. \end{cases}$$

It is not difficult to see that $\varphi \in \alpha(G, k)$ and φ is persistent-interval on R_0 .

Corollary 8 [17] *Let G be a bipartite $(3, 4)$ -biregular graph. Then there exists a coloring from $\alpha(G, 4)$ which is persistent-interval for at least $\frac{4}{7}|V(G)|$ vertices of G .*

3. Acknowledgment

The author thanks professors A.S. Asratian and P.A. Petrosyan for their attention to this work.

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